THE CENTRAL LIMIT THEOREM FOR GEODESIC FLOWS ON n-DIMENSIONAL MANIFOLDS OF NEGATIVE CURVATURE

BY

M. RATNER

ABSTRACT

In this paper we prove a central limit theorem for special flows built over shifts which satisfy a uniform mixing of type γ^{n^α}, $0 < \gamma < 1$, $\alpha > 0$. The function defining the special flow is assumed to be continuous with modulus of continuity of type $\rho^{|\log d(x_1, x_2)|^\beta}$, $0 < \rho < 1, \beta > 0$, and d is the natural metric on sequence space. Geodesic flows on compact manifolds of negative curvature are isomorphic to special flows of this kind.

DEFINITION. Let f be a measurable, bounded real function, defined on a Lebesgue space M with measure m. f is said to satisfy the central limit theorem relative to a measurable ergodic flow $\{S^t\}$ in M if there exists a constant $\sigma > 0$ such that for any $-\infty < \alpha < \infty$

$$(1) \qquad \lim_{t\to\infty} m\left\{x: \int_0^t (f(S^\tau x) - \bar{f})d\tau / \sigma\sqrt{t} < \alpha\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}u^2} du$$

where $\bar{f} = \int_M f(x)dm$.

An analogous definition holds for automorphisms; the only change is to replace the integral by a sum.

Sinai [12] proved the central limit theorem for a wide class of functions for the case of a geodesic flow in a space of linear elements of a compact manifold M of constant negative curvature. The study of this class in [12] makes essential use of the properties of M as a homogeneous space and of the representation of its group of motions. These methods do not apply to the case of varying curvature. This case was considered for three-dimensional compact manifolds in [9]. The

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central limit theorem (*clt*) was proved there for arbitrary Anosov flows (which we shall henceforth call C-flows) of class C^2 relative to a smooth invariant measure (see [13], [10], [14]), which is simply invariant Riemannian volume in the case of geodesic flows.

In this paper we prove the *clt* for transitive Anosov flows of class C^2 on compacts Riemannian manifolds M of any dimension. The proof makes essential use of a special representation of a flow $\{T^t\}$ obtained by means of a Markov partition (see [13], [2], [3], [11]). This partition determines a matrix $\pi = \| \pi_{ij} \|$, $\pi_{ij} = 0, 1$, of order r, such that for some integer $s > 0$ the elements of the matrix π^s are positive. Using this matrix, we then construct the space $X_\pi = X \subset \{1, 2, \cdots, r\}^Z$ of sequences $x = \{x_i\}_{i=-\infty}^{\infty}$, $\pi_{x_i, x_{i+1}} = 1$, with the metric

$$\rho(x', x'') = \Sigma\; 2^{-|i|} e(x_i', x_i''), \text{ where}$$

$$e(x_i', x_i'') = \begin{cases} 0 & x_i' = x_i'' \\ 1 & x_i' \neq x_i''. \end{cases}$$

The space X is the domain of the shift automorphism ϕ: $(\phi x)_i = x_{i-1}$ (see [8]). The Markov partition enables us to define: (i) a continuous positive function $l(x)$ on X satisfying a Holder condition; (ii) a special flow S^t acting in the space $W = (X, l) = \{(x, y) : x \in X,\ 0 \leqq y < l(x),\ (x, l(x)) = (\phi x, 0)\}$ with the direct product metric, so that for $t < \inf_{x \in X} l(x)$,

$$S^t(x, y) = \begin{cases} (x, y + t) & t < l(x) - y \\ (\phi x, t + y - l(x)) & t \geqq l(x) - y \end{cases}$$

and S^t is uniquely determined for other values of t by the condition that it be a one-parameter transformation group; (iii) a continuous mapping ψ: $W \to M$ such that $\psi S^t = T^t \psi$.

Now, if ν is an S^t-invariant Borel measure in W such that the set on which ψ fails to be one-to-one has ν-measure 0, then the flows S^t in (W, ν) and T^t in $(M, \psi * \nu)$ are isomorphic (for a Borel set $A \subset M$, $\psi * \nu(A) = \nu(\psi^{-1} A)$).

This was precisely the method used by Sinai in [14] to construct invariant Gibbs measures for transitive C-flows of class C^2. A Gibbs measure ν in W induces a ϕ-invariant measure μ on X such that $d\nu = (d\mu \times dt)(1/\bar{l})$, where $\bar{l} = \int_X l(x) d\mu$ and the shift ϕ in (X, μ) is a K-automorphism with a strong mixing of type $\Upsilon_{\gamma, \alpha}$, $0 < \gamma < 1$, $\alpha > 0$ (see [8], [10], [14]), that is, for any sets $B_i \in \mathcal{M}_{k+n}^{\infty}$, $B_i \cap B_j = \phi$ $(i \neq j)$ $A \in \mathcal{M}_{-\infty}^{k}$,

(2) $$\sum_i \left|\mu(B_i/A) - \mu(B_i)\right| < C\gamma^{n^\alpha}.$$

$\mathcal{M}_a^b$ is the σ-algebra of the sets measurable with respect to $\{x_i|_{i=a}^b\}$ and $C > 0$ is a constant. The function l is assumed to be of class $\Upsilon_{\rho,\kappa}$, that is, if $(x')_i = (x'')_i$ for $|i| \leqq n$, then

$$\left|l(x') - l(x'')\right| \leqq A\rho^{n^\kappa}$$

for constants $A = A(l) > 0$, $0 < \rho < 1$, $\kappa > 0$.

Our main result is the *clt* for a wide range of continuous functions in W relative to the flow S^t in (W, ν) with condition (2) and a function $l(x) \in \Upsilon_{\rho,\kappa}$.

Since smooth invariant measures for transitive C-flows of class C^2 are Gibbs measures [14], the main result implies the *clt* for such measures, in particular, the *clt* for geodesic flow on manifolds of negative curvature relative to invariant Riemannian volume. The class of functions for which the *clt* holds coincides with the class of functions found in [11] for constant curvature.

1. Auxiliary lemmas

Let ϕ be the shift automorphism in (X, μ) with condition (2).

LEMMA 1.1. *Let $F \in \Upsilon_{\rho,\kappa}$ on X and $D_N F \to \infty$ as $N \to \infty$, where*

$$D_N(F) = \int_X \left[\sum_{i=1}^N (F(\phi^{-i}x) - \bar{F})\right]^2 d\mu$$

$$\bar{F} = \int_X F(x)d\mu = E(F).$$

Then $D_N F \sim \sigma_F N$, $\sigma_F > 0$, and F satisfies the clt; moreover, $\sigma = \sqrt{\sigma_F}$ in (1).

PROOF. For $x \in X$, we set

$$\Delta_{-k}^k(x) = \{x' \in X : x_i' = x_i \,|\, |i| \leqq k\}$$

and denote

$$F_k(x) = \int_{\Delta_{-k}^k(x)} F(x')d\mu_{\Delta_{-k}^k(x)}$$

where the integration is with respect to the conditional measure induced by μ on $\Delta_{-k}^k(x)$. Since $F \in \Upsilon_{\rho,\kappa}$, it follows that in the $L_\mu^2(X)$-norm

(3) $$\left\| F(x) - F_k(x) \right\| < A\rho^{k^\kappa}.$$

It then follows from [6] that when condition (2) holds, $D_N F \sim \sigma_F N$ for $\sigma_F > 0$, as $N \to \infty$, and the function F satisfies the *clt*. ∎

Likewise it follows from [6] (see also [1]) that if $D_k F_{[k^\delta]} \sim Ck$ as $k \to \infty$ for $0 < \delta < 1$, where $C > 0$ is a constant, then for some $\tau = \tau(\delta) > 0$:

$$\left| E\left(\exp\left\{ iz \frac{\sum_{i=0}^{k} (F_{[k^\delta]}(\phi^{-i}x) - \bar{F})}{\sqrt{D_k F_{[k^\delta]}}} \right\}\right) - \exp\{-\tfrac{1}{2}z^2\} \right| \leqq 1/k^\tau \tag{4}$$

for $z \in [-k^\tau, k^\tau]$.

The question of conditions on F under which $D_N F \sim \sigma_F N$, $\sigma_F > 0$ is studied in [7]. (According to our assumptions, if $F \in \Upsilon_{\rho,\kappa}$ this is equivalent to $D_N F \to \infty$ as $N \to \infty$.)

Let U be the unitary operator in $L^2_\mu(X)$ adjoint to ϕ. Every function $F \in L^2_\mu(X)$ has an absolutely continuous spectral function relative to U. In this case, either $D_N F \to \infty$ or $D_N F < c < \infty$. Let $r_F(\rho)$ be the spectral density of F. It was shown in [7] that if (i) $r_F(\rho)$ is continuous at $\rho = 0$ and (ii) $r_F(0) = r_0 > 0$, then $D_N F \sim 2\pi r_0 N$ as $N \to \infty$.

It follows from (3) and condition (2) that the correlation function of $F \in \Upsilon_{\rho,\kappa}$ decreases to zero at a rate of type $\rho_1^{n^{\alpha_1}}$, $0 < \rho_1 < 1$, $\alpha_1 > 0$. In this case [7] conditions (i)–(ii) are surely satisfied when the equation $UG - G = F - \bar{F}$ has no solutions in $L^2_\mu(x)$. But if there is a solution in $L^2_\mu(X)$, then the variance $D_N(F)$ is bounded.

Now let $l \in \Upsilon_{\rho,\kappa}$ be a positive function on X and S^t the special flow in (W, ν) constructed over (X, μ) with the aid of the function l, $d\nu = (d\mu \times dt)/\bar{l}$. It is assumed that S^t is a K-flow in (W, ν) (this is true in the case of Gibbs measures of transitive C-flows). It then follows from [5] that the equation $UG - G = l - \bar{l}$ has no solutions in $L^2_\mu(X)$, since the existence of such a solution would imply that the spectrum of the flow S^t has a discrete component. Thus l satisfies the *clt*.

Lemma 1.2. *Let $F \in \Upsilon_{\rho,\kappa}$, $K \in \Upsilon_{\rho_1,\kappa_1}$ be continuous on X and $D_n F \sim \sigma_F n$ ($\sigma_F > 0$). Then*

$$\lim_{n\to\infty} E\left(K(x) \exp\left\{ iz \frac{\sum_{i=0}^{n} F(\phi^{-i}x) - n\bar{F}}{\sqrt{\sigma_F n}} \right\}\right) = \bar{K} \exp(-\tfrac{1}{2}z^2) . \tag{5}$$

The convergence is uniform in z on every finite interval.

Proof. We write the sum in (5) as

$$\sum_{i=0}^{[n^{\frac{1}{4}}]-1} (F(\phi^{-i}x) - \bar{F}) + \sum_{i=[n^{\frac{1}{4}}]}^{n} (F(\phi^{-i}x) - \bar{F}) = J_1 + J_2.$$

Since F is bounded on X, it follows that for some constant $C_1 > 0$

$$\left|\frac{J_1}{\sqrt{\sigma_F n}}\right| < C_1 n^{-\frac{1}{4}}.$$

Therefore,

$$\left|\exp\left\{iz\frac{J_1+J_2}{\sqrt{\sigma_F n}}\right\} - \exp\left\{iz\frac{J_2}{\sqrt{\sigma_F n}}\right\}\right| \leqq r_n'$$

where r_n' is independent of x and $r_n' \to 0$ as $n \to \infty$, uniformly in z on every finite interval.

Let $0 < \delta < \frac{1}{4}$; consider the function $F_{[n^\delta]}(x)$. Then, setting $H_{[n^\delta]}(x) = F(x) - F_{[n^\delta]}(x)$, we conclude from (3) that for all $x \in X$

$$|H_{[n^\delta]}(x)| < A\rho^{n^{\delta\kappa}}.$$

Consider the sum

$$J_2 = \sum_{i=n^{\frac{1}{4}}}^{n} (F(\phi^{-i}x) - \bar{F}) = \sum_{i=n^{\frac{1}{4}}}^{n} (F_{[n^\delta]}(\phi^{-i}x) - \bar{F}) +$$

$$+ \sum_{i=n^{\frac{1}{4}}}^{n} H_{[n^\delta]}(\phi^{-i}x) = I_1 + I_2,$$

where

$$\left|\frac{I_2}{\sqrt{\sigma_F n}}\right| < An\rho^{n^\delta}.$$

Then

$$\left|\exp\left\{iz\frac{J_2}{\sqrt{\sigma_F n}}\right\} - \exp\left\{iz\frac{I_1}{\sqrt{\sigma_F n}}\right\}\right| < r_n''$$

where r_n'' is independent of x and $r_n'' \to 0$ as $n \to \infty$, uniformly in z on every finite interval. By (3), $D_n(F_{[n^\delta]}) \sim n\sigma_F$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} E\left(K(x)\exp\left\{iz\sum_{i=1}^{n}(F_{[n^\delta]}(\phi^{-i}x) - \bar{F})/\sqrt{\sigma_F n}\right\}\right)$$

$$= \lim_{n\to\infty} E\left(K_{[n^\delta]}(x)\exp\left\{iz\sum_{i=n^{\frac{1}{4}}}^{n}(F_{[n^\delta]}(\phi^{-i}x) - \bar{F})/\sqrt{D_n F_{[n^\delta]}}\right\}\right).$$

It follows from condtion (2) that

$$\left| E\left(K_{[n^\delta]}(x)\exp\left\{iz\sum_{i=n^{\frac{1}{4}}}^{n}(F_{[n^\delta]}(\phi^{-i}x) - \bar{F})/\sqrt{D_n F_{[n^\delta]}}\right\}\right) - \right.$$

$$\left. - \bar{K}E\left(\exp\ iz\sum_{i=n^{\frac{1}{4}}}^{n}(F_{[n^\delta]}(\phi^{-i}x) - \bar{F})/\sqrt{D_n F_{[n^\delta]}}\right\}\right)\right| < r_n'''$$

where $r_n''' \to 0$ as $n \to \infty$ uniformly in z on every finite interval. The assertion now follows easily from (4). ■

COROLLARY 1.3. *Let $R \in \Upsilon_{\rho,\kappa}$, $Q \in \Upsilon_{\rho_1,\kappa^1}$ be continuous on X and $D_nR \sim \sigma_R n$, $\sigma_R > 0$, $D_nQ \sim \sigma_Q n$, $\sigma_Q > 0$. In Lemma 1.2, set $K(x) = l(x)$ (the special representation function) and $F(x) = z_1R(x) + z_2Q(x)$, where z_1, z_2 are arbitrary real numbers. Then, setting $z = 1$ in (5), we get*

$$\lim_{n\to\infty} (1/\bar{l})E\left(l(x)\exp\left\{iz_1 \sum_{i=1}^{n} (R(\phi^{-i}x) - \bar{R})/\sqrt{n} + iz_2 \sum_{i=1}^{n} (Q(\phi^{-i}x)-\bar{Q})/\sqrt{n}\right\}\right)$$

$$= \exp\{-\tfrac{1}{2}(z_1^2\sigma_R + 2b_{RQ}\, z_1 \cdot z_2 + z_2^2\sigma_Q)\} \tag{6}$$

where

$$b_{RQ} = \lim_{n\to\infty} \left\{E\left(\sum_{i=1}^{n} (R(\phi^{-i}x) - \bar{R}) \cdot \sum_{i=1}^{n} (Q(\phi^{-i}x) - \bar{Q})\right)\Big/ n\right\}.$$

Indeed, if z_1 and z_2 are such that $D_n(z_1R + z_2Q) \sim dn$, $d > 0$, then (6) follows at once from (5). But if z_1 and z_2 are such that the variance $D_n(z_1R + z_2Q)$ is bounded as $n \to \infty$, this means that the limit distribution is degenerate; but then also $z_1^2\sigma_R + 2b_{RQ}z_1z_2 + z_2^2\sigma_Q = 0$, and so (6) remains valid.

If we let μ_l denote the measure on X defined by $d\mu_l = (l(x)\,/\,\bar{l})d\mu$, then (6) means that the two-dimensional *clt* is satisfied with respect to the measure μ_l.

We now consider the special flow S^t in $(W,\nu) = (X,\mu,l)$. We shall adopt the convention that lower case Latin letters denote functions on W; upper case Latin letters denote functions on X. If $f(w)$ and $F(x)$ are functions on (W,ν) and (X,μ), respectively, then $N(f)$ and $E(F)$ will denote their means:

$$\bar{f} = N(f) = \int_W f(w)d\nu; \; \bar{F} = E(F) = \int_X F(x)d\mu.$$

For $w \in W$, we write $w = (x,y)$, where $x \in X$ and $0 \leqq y < l(x)$. With any function $f(w)$ on W we associate a function $F(x)$ on X as follows:

$$F(x) = \int_0^{l(x)} f(x,y)dy.$$

Let V be the infinitesimal operator corresponding to the group $\{V_t\}$ of unitary operators adjoint to the flow S^t, that is, $V_t = \exp(itV)$. Let $f \in L^2_\nu(W)$, and consider the following equations: in $L^2_\nu(W)$,

$$Vh(w) = f(w) - \bar{f} \tag{7}$$

and in $L^2_\mu(X)$,

$$UH(x) - H(x) = F(x) - (\bar{F}/\bar{l})l(x). \tag{8}$$

It is obvious that $\bar{f} = \bar{F}/\bar{l}$. $UH(x) = H(\phi x)$.

LEMMA 1.4. *Equation* (7) *is solvable iff equation* (8) *is solvable.*

PROOF. Assume that $h(w) \in L_\nu^2(W)$ satisfies equation (7). Then

$$\int_0^{l(x)} Vh(x,y)dy = \int_0^{l(x)} (f(x,y) - \bar{f})dy = F(x) - (\bar{F}/\bar{l})l(x).$$

It is readily shown that the following formula is valid in $L_\nu^2(W)$:

$$\int_0^{l(x)} Vh(x,y)dy = h(x,l(x)) - h(x,0);\ h(x,0) \in L_\mu^2(X).$$

But $h(x,l(x)) = h(\phi x, 0) = Uh(x,0)$. Therefore the function $H(x) = h(x,0) \in L_\mu^2(X)$ satisfies equation (8).

Now let $H(x)$ satisfy equation (8), that is,

$$UH(x) - H(x) = \int_0^{l(x)} (f(x,y) - \bar{f})dy,\ H(x) \in L_\mu^2(X).$$

Consider the function

$$h(x,y) = H(x) + \int_0^y (f(x,z) - \bar{f})dz.$$

Then $h(x, l(x)) = h(\phi x, 0)$. Therefore $h(x,y) = h(w) \in L_\nu^2(W)$ and $h(w)$ satisfies equation (7). ∎

2. The *clt* for the special flow

It is assumed here that $l \in \Upsilon_{\rho,\kappa}$ and $D_n l \sim \sigma_l n$, $\sigma_l > 0$ (as shown above, this is the case, for example, if S^t is a K-flow in (W,ν)). Then l satisfies the *clt.*

We shall say that $f \in \Upsilon_{\rho,\kappa}$ on W if

$$F(x) = \int_0^{l(x)} f(x,y)\,dy \in \Upsilon_{\rho,\kappa} \text{ on } X.$$

THEOREM 2.1 *Let $f \in \Upsilon_{\rho,\kappa}$ be continuous on W and suppose that equation* (7) *has no solution in $L_\nu^2(W)$. Then f satisfies the clt relative to S^t, and moreover*

$$\sigma^2 = (2\pi/\bar{l})r_{F(x)-(\bar{F}/\bar{l})l(x)}(0) > 0$$

in (1), *where $r_G(\rho)$ is the spectral density of G.*

PROOF. Since $\tilde{F}(x) = F(x) - (\bar{F}/\bar{l})\ l(x) \in \Upsilon_{\rho,\kappa}$ it follows from Lemma 1.4 that

$D_n(\tilde{F}) \sim \sigma_{\tilde{F}} n$ as $n \to \infty$, where $\sigma_{\tilde{F}} = 2\pi r_{\tilde{F}}(0) > 0$. Then, by Lemma 1.1 $\tilde{F}$ satisfies the *clt* relative to ϕ in (X, μ).

Define a function $n(t, x)$ by

$$\sum_{i=0}^{n(t,x)} l(\phi^{-i}x) < t \leqq \sum_{i=0}^{n(t,x)+1} l(\phi^{-i}x).$$

In other words, $n(t,x)$ is the number of times the trajectory of the flow S^t, issuing from x in the negative direction, hits X during time t. Since $l(x)$ satisfies the *clt*, one easily infers (see, for example, [4]) that for any fixed z, $-\infty < z < \infty$,

$$\lim_{t \to \infty} \mu \left\{ x \colon \frac{n(t,x) - t/\bar{l}}{\sigma_l \sqrt{t}(\bar{l})^{-3/2}} < z \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du.$$

For $w = (x, y)$, we denote

$$a(t,w) = \left(\int_0^t f(S^{-u}w) du - t\bar{f} \right) / \sqrt{t}$$

$$B(t,x) = \left(\int_0^t f(S^{-u}(x,0))\, du - t\bar{f} \right) / \sqrt{t}.$$

It is clear that

$$\left| a(t,w) - B(t,x) \right| < C_1 / \sqrt{t}$$

where $C_1 > 0$ is a constant independent of w and x.

Then, for any z in a finite interval $[-K, K]$,

$$\left| N(\exp\{iza(t,w)\}) - \frac{1}{\bar{l}} \int_X d\mu \int_0^{l(x)} \exp\{izB(t,x)\} dy \right| < \frac{C_1 K}{\sqrt{t}}. \tag{9}$$

We have

$$\frac{1}{\bar{l}} \int_X d\mu \int_0^{l(x)} \exp\{izB(t,x)\} dy = \frac{1}{\bar{l}} E(l(x) \exp\{izB(t,x)\}) = E_{\mu_l}(\exp\{izB(t,x)\}).$$

Let $\varepsilon > 0$ be arbitrary and A_{kt} the set

$$A_{kt} = \left\{ x \in X \colon \frac{t}{\bar{l}} + k\varepsilon\sqrt{t} \leqq n(t,x) < \frac{t}{\bar{l}} + (k+1)\varepsilon\sqrt{t} \right\}, \qquad \bigcup_{k=-\infty}^{\infty} A_{kt} = X.$$

Then:

$$E(l(x)\exp\{izB(t,x)\}) = \sum_{k=-\infty}^{\infty} \int_{A_{kt}} l(x) \exp\{izB(t,x)\} d\mu.$$

Define L by

$$\frac{1}{\sqrt{2\pi}}\int_{|u|>L} e^{-\frac{1}{2}u^2}\,du \leqq \frac{\varepsilon}{2}.$$

Then there exists t_0 such that for all $t \geqq t_0$

$$\mu\left\{x : \frac{|n(t,x) - t/\bar{l}|}{\sqrt{t}} > L\right\} < \varepsilon. \tag{10}$$

Consider the sets A_{kt} for $|k\varepsilon| \leqq L$. On these sets, we have

$$\left| B(t,x) - \frac{\sum_{i=0}^{n(t,x)} F(\phi^{-i}x) - (\bar{F}/\bar{l}) \sum_{i=0}^{n(t,x)} l(\phi^{-i}x)}{\sqrt{\bar{l}n(t,x)}}(1 + b_t^1(x)) \right| < \frac{C_1}{\sqrt{t}} \tag{11}$$

where $|b_t^1(x)| \leqq b_t^1$ and $b_t^1 \to 0$ as $t \to \infty$, uniformly in $|k\varepsilon| \leqq L$.

We denote

$$G(t,x) = \left(\sum_{i=0}^{n(t,x)} \tilde{F}(\phi^{-i}x)\right) / \sqrt{\bar{l}n(t,x)}$$

where $\tilde{F}(x) = F(x) - (\bar{F}/\bar{l})\, l(x)$.
It follows from (9), (10), and (11) that for $z \in [-K, K]$

$$\left| N(\exp\{iza(t,w)\}) - \frac{1}{\bar{l}} \sum_{k\varepsilon=-L}^{L} \int_{A_{kt}} l(x) \exp\{izG(t,x)(1 + b_t'(x))\} d\mu \right| \tag{12}$$
$$< \frac{C_1 K}{\sqrt{t}} + \varepsilon.$$

For $x \in A_{kt}$, we set $\tilde{n}(t,x) = n(t,x) - (t/\bar{l} + k\varepsilon\sqrt{t})$. We rewrite $G(t,x)$ thus:

$$G(t,x) = \left\{ \sum_{i=0}^{[t/\bar{l}+k\varepsilon\sqrt{t}]} \tilde{F}(\phi^{-i}x) + \sum_{i=[t/\bar{l}+k\varepsilon\sqrt{t}]+1}^{n(t,x)} \tilde{F}(\phi^{-i}x) \right\} \Big/ \left(\bar{l}(t/\bar{l} + k\varepsilon\sqrt{t} + \tilde{n}(t,x)\right)^{\frac{1}{2}}.$$

Since $|\tilde{n}(t,x)| \leqq \varepsilon\sqrt{t}$ for $x \in A_{kt}$, it follows that

$$G(t,x) = \frac{\sum_{i=0}^{[t/\bar{l}+k\varepsilon\sqrt{t}]} \tilde{F}(\phi^{-i}x)}{[\bar{l}(t/\bar{l} + k\varepsilon\sqrt{t})]^{\frac{1}{2}}}(1 + b_t^3(x)) + b_t^2(x)$$

where $|b_t^2(x)| \leqq \varepsilon b$, $b > 0$ a constant, $|b_t^3(x)| \leqq b_t^3 \to 0$ as $t \to \infty$ uniformly in $|k\varepsilon| \leqq L$.

We denote

$$H^k(t,x) = \left(\sum_{i=0}^{[t/l+k\varepsilon\sqrt{t}]} \tilde{F}(\phi^{-i}x)\right) \Big/ \left(\bar{l}(t/\bar{l}+k\varepsilon\sqrt{t})\right)^{\frac{1}{2}}.$$

Then, in view of (12), we have for $z \in [-K, K]$

$$\left| N(\exp\{iza(t,w)\}) - \frac{1}{\bar{l}} \sum_{k\varepsilon=-L}^{L} \int_{A_{kt}} l(x) \exp\{izH^k(t,x)(1+b_t^3(x))(1+b_t^1(x))\} d\mu \right|$$

(13)

$$< \frac{C_1 K}{\sqrt{t}} + \varepsilon + \varepsilon K b.$$

We now study the sets A_{kt} more closely. They are defined by

$$A_{kt} = \left\{ x: \sum_{i=0}^{[t/l+k\varepsilon\sqrt{t}]} l(\phi^{-i}x) < t \leqq \sum_{i=0}^{[t/l+(k+1)\varepsilon\sqrt{t}]} l(\phi^{-i}x) \right\}.$$

The sum on the right is

$$\sum_{i=0}^{[t/\bar{l}+(k+1)\varepsilon\sqrt{t}]} l(\phi^{-i}x) = \sum_{i=0}^{[t/\bar{l}+k\varepsilon\sqrt{t}]} l(\phi^{-i}x) + \sum_{i=[t/\bar{l}+k\varepsilon\sqrt{t}]+1}^{[t/\bar{l}+(k+1)\varepsilon\sqrt{t}]} l(\phi^{-i}x) = I_1(x) + I_2(x).$$

Since ϕ is ergodic, it follows that for δ_1, $\delta_2 > 0$ there exists $t_1 > 0$ such that for $t \geqq t_1$

$$\mu\{x: |I_2(x) - \varepsilon\sqrt{t}| \leqq \varepsilon\sqrt{t}\delta_1\} \geqq 1 - \delta_2. \tag{14}$$

Let $A'_{kt} \subset A_{kt}$ denote the set of all $x \in A_{kt}$ for which (14) holds, and set $I_2(x) - \varepsilon\sqrt{t}\,\bar{l} = \varepsilon\sqrt{t}\,\delta_1(x)$. Then

$$A'_{kt} = \left\{ x: t - \varepsilon\sqrt{t}\,\bar{l} - \varepsilon\sqrt{t}\,\delta_1(x) \leqq \sum_{i=0}^{[t/\bar{l}+k\varepsilon\sqrt{t}]} l(\phi^{-i}x) < t \right\}$$

$$= \left\{ x: \frac{-(k+1)\varepsilon}{(1/\bar{l}+k\varepsilon/\sqrt{t})^{\frac{1}{2}}} - \delta_1(x)\varepsilon \leqq \frac{\sum_{i=0}^{[t/\bar{l}+k\varepsilon\sqrt{t}]} (l(\phi^{-i}x) - \bar{l})}{(t/\bar{l}+k\varepsilon\sqrt{t})^{\frac{1}{2}}} < \frac{-k\varepsilon}{(1/\bar{l}+k\varepsilon/\sqrt{t})^{\frac{1}{2}}} \right\}$$

where $|\delta_1(x)| < \delta_1$, $\mu(A_{kt} \Theta A'_{kt}) < \delta_2$, δ_1, $\delta_2 \to 0$ as $t \to \infty$ for $|k\varepsilon| \leqq L$.

We denote

$$A''_{kt} = \left\{ x: \frac{-(k+1)\varepsilon}{(1/\bar{l}+k\varepsilon/\sqrt{t})^{\frac{1}{2}}} \leqq \frac{\sum_{i=0}^{[t/\bar{l}+k\varepsilon\sqrt{t}]} (l(\phi^{-i}x) - \bar{l})}{(t/\bar{l}+k\varepsilon\sqrt{t})^{\frac{1}{2}}} < \frac{-k\varepsilon}{(1/\bar{l}+k\varepsilon/\sqrt{t})^{\frac{1}{2}}} \right\}.$$

It is clear that $\mu(A_{kt} \Theta A''_{kt}) \to 0$ as $t \to \infty$ uniformly in $|k\varepsilon| \leqq L$.

Thus, we can replace the set A_{kt} in (13) by the set A''_{kt} defined by the sum

$$\xi^k(t,x) = \sum_{i=0}^{[t/l+k\varepsilon\sqrt{t}]} (l(\phi^{-i}x) - \bar{l})/(t/\bar{l} + k\varepsilon\sqrt{t})^{\frac{1}{2}}.$$

Consider the pair of random variables $(\xi^k(t,x),\ H^k(t,x))$. We know that $D_n(l) \sim \sigma_l n$ and $D_n(F - (\bar{F}/\bar{l})l) \sim \sigma_F n$. Applying Corollary 1.3, we see that for fixed k and $t \to \infty$ the two-dimensional distribution of the vector $(\xi^k(t,x),\ H^k(t,x))$ is asymptotically normal with respect to the measure μ_l, with covariance matrix $\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}$, where

$$\alpha = \sigma_l,$$

$$\gamma = \frac{1}{\bar{l}}\sigma_F \qquad \beta = b_{l,F/\sqrt{l}}\,.$$

Therefore, for fixed ε,

$$\Big|\frac{1}{\bar{l}}\sum_{k\varepsilon=-L}^{L}\int_{A_{kt}} l(x)\exp\{izH^k(t,x)(1+b_t^3(x))(1+b_t'(x))\}d\mu \; - $$

$$- \sum_{k\varepsilon=-L}^{L}\iint_{-(k+1)\varepsilon\sqrt{l}\leqq u_1\leqq -k\varepsilon\sqrt{l}} \exp\{izu_2)\Phi(du_1,du_2)\Big| \to 0$$

as $t \to \infty$, where $\Phi(du_1, du_2)$ is the two-dimensional normal distribution with covariance matrix $\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}$ and zero expectation.

It then follows from (13) that

$$\overline{\lim_{t\to\infty}}\Big(N(\exp\{iza(t,w)\}) - \sum_{k\varepsilon=-L}^{L}\iint_{-(k+1)\varepsilon\sqrt{l}\leqq u_1\leqq -k\varepsilon\sqrt{l}} \exp\{izu_2\}\,\Phi(du_1,du_2)\Big|$$

$$\leqq (1+K)\varepsilon$$

for $z \in [-K, K]$.

But for $\varepsilon \to 0$, $L \to \infty$, it is also true that

$$\Big|\sum_{k\varepsilon=-L}^{L}\iint_{-(k+1)\varepsilon\sqrt{l}\leqq u_1\leqq -k\varepsilon\sqrt{l}} \exp\{izu_2\}\Phi(du_1,du_2)$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{izu_2)\,\Phi(du_1,du_2)\Big| \to 0.$$

Then

$$\lim_{t\to\infty} N(\exp\{iza(t,w)\}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{izu_2\}\,\Phi(du_1,du_2) = \exp\Big\{-\frac{z^2}{2\sigma_F/\bar{l}}\Big\}.$$

But $\sigma_F = 2\pi r_F(0)$. This completes the proof of Theorem 2.1. ∎

REMARK (i). If ν is a Gibbs measure in W, then μ is a Gibbs measure in X (see [14]) and satisfies condition (2). Under these conditions, if S^t is a K-flow in (W, ν), Theorem 2.1 is the *clt* for Gibbs measures.

REMARK (ii). Let T^t be a transitive C-flow of class C^2 on (M, ν^*) with Gibbs measure ν^*. As stated above, T^t is a K-flow in (M, ν^*) (see [14]) and is isomorphic to the special flow S^t in (W, ν) with Gibbs measure ν. Moreover, it was shown in [14] (see also [11]) that this isomorphism $\psi: W \to M$ is such that, if $h \in \Upsilon_{\rho,\kappa}$ on M, that is, $|h(z) - h(z')| < C\rho^{|\log d(z,z')|^\kappa}$ for some $C, \kappa > 0$, $0 < \rho < 1$, d the metric in M, then the function $f(w) = h(\psi w)$ belongs to $\Upsilon_{\rho_1,\kappa}$ on W with $0 < \rho_1 < 1$. Thus Theorem 2.1 is the *clt* for the class of functions $h \in \Upsilon_{\rho,\kappa}$ relative to a transitive C-flow of class C^2 on M. For geodesic flows on a manifold of negative curvature, this class of functions is precisely that found in [12] for the case of constant curvature.

3. Asymptotic behavior of variance

We shall show here that for $f \in \Upsilon_{\rho,\kappa}$ on (W, ν) the normalizing factor in the *clt* is simply the variance $D_t f$, that is, we shall prove Theorem 3.1.

THEOREM 3.1. *Let $f \in \Upsilon_{\rho,\kappa}$ and suppose that equation (7) has no solution in $L^2_\nu(W)$. Then $D_t f \sim \sigma_f t$ as $t \to \infty$, where*

$$\sigma_f = \frac{2\pi}{l} r_F(0) > 0.$$

In the opposite case the variance $D_t f$ is bounded as $t \to \infty$.

LEMMA 3.2. *Let $F \in \Upsilon_{\rho,\kappa}$ on (X, μ), $\bar{F} = 0$, $S^n_F = \sum_{i=1}^n F(\phi^{-i}x)$. For any integer $r > 0$,*

$$E(S^n_F)^{2r} = \int_X [S^n_F(x)]^{2r} d\mu \leqq C_r n^r$$

where $C_r > 0$ is a constant depending only on r.

PROOF. We confine ourselves to the case $r = 3$. For other values of r the proof is analogous.

We have

$$(15) \qquad E(S^n_F)^6 = \sum_{k_1,\cdots,k_6} E(F(\phi^{-k_1}x)\cdots F(\phi^{-k_6}x))$$

where k_j, $j = 1, \cdots, 6$, take values from 1 to n. Let $k = (k_1, \cdots, k_6)$ and $i = (i_1, \cdots, i_6)$ be two sextuples of integers, and set

$$e(k,i) = \max(|k_1 - i_1|, \cdots, |k_6 - i_6|).$$

Let A denote the set of sextuples $(k_1, \cdots, k_6) = k$, $1 \leqq k_l \leqq n$, such that for any k_l there exists $k_j = k_l$ for some $j \neq l$. Then the sum in (15) can be written

$$E(S_F^n)^6 = \sum_{k \in A} + \sum_{k:1 \leqq e(k,A) \leqq 2} + \cdots + \sum_{k:2^i \leqq e(k,A) \leqq 2^{i+1}} + \cdots. \tag{16}$$

In any sextuple k in the ith sum, there exists k_j, $1 \leqq j \leqq 6$, such that $|k_j - k_l| \geqq 2$ for all $l \neq j$. For such sextuples it follows from (2) and (3) that

$$|E(F(\phi^{-k_1}x) \cdot F(\phi^{-k_2}x) \cdots F(\phi^{-k_6}x))| \leqq C\lambda^{2^{i\alpha}} \tag{17}$$

where $C, \alpha > 0$ are constants and $0 < \lambda < 1$. Let us estimate the number of terms in the ith sum. Let $m(A)$ denote the number of sextuples in A. It is clear that the number of sextuples k such that $e(k,A) < 2^i$ does not exceed the number $m(A) \cdot (2^i + 1)^6$. In order to estimate $m(A)$, we observe that the sextuples in A may be divided into four types: (i) three distinct pairs of equal numbers; (ii) a quadruple and a pair of equal numbers; (iii) two triples of equal numbers; and (iv) all six numbers equal. The number of sextuples of the first type is at most $C_1 n^3$, of the second and third types $C_2 n^2$, and of the fourth type $C_3 n$. Therefore $m(A) \leqq C_4 n^3$. Thus, in view of the fact that F is bounded on X, we obtain from (16) and (17)

$$E(S_F^n)^6 \leqq C_5 n^3 \left(1 + \sum_{i=0}^{\infty} 2^{6i} \lambda^{2^{\alpha i}}\right) \leqq C n^3$$

where C, $C_i > 0$, $i = 1, \cdots, 5$, are constants.

This completes the proof. ∎

We now estimate the integral $\int_{|z| \leqq K} z^i d\Phi(z)$, where $\Phi(z)$ is the distribution of S_F^n, for any even $i > 0$.

LEMMA 3. 2. *For* $0 < i < 2r$,

$$\int_{|z| \leqq K} z^i d\Phi(z) \leqq \tilde{C}_r n^r / K^{2r-i}$$

where $\tilde{C}_r > 0$ *is a constant depending only on i and r.*

PROOF. Integrating by parts and using Chebyshev's inequality and Lemma 3.2, we have

$$\int_{|z| > K} z^i d\Phi(z) = \int_{-\infty}^{-K-0} z^i d\Phi(z) + \int_{K+0}^{\infty} z^i d(\Phi(z) - 1) = z^i \Phi(z) \Big|_{-\infty}^{-K-0}$$

$$- \int_{-\infty}^{-K-0} \Phi(z) \cdot i z^{i-1} dz + z^i(\Phi(z) - 1) \Big|_{K+0}^{\infty} - \int_{K+0}^{\infty} [\Phi(z) - 1] i z^{i-1} dz$$

$$= K^i\Phi(-K-0) - i\int_{-\infty}^{-K-0}\Phi(z)z^{i-1}dz + K^i(\Phi(K+0)-1) -$$

$$- i\int_{K+0}^{\infty}[\Phi(z)-1]z^{i-1}dz \leqq K^i\frac{E(S_F^n)^{2r}}{K^{2r}} + iE(S_F^n)^{2r}\cdot\int_{K+0}^{\infty}z^{i-1-2r}dz$$

$$\leqq C_r n^r K^{-2r+i} + i\, C_r n^r K^{-2r+i} = \tilde{C}_r n^r K^{-2r+i}.$$

This proves the lemma. ∎

We now consider the random variable $n(t,x)$ of Section 2. For this variable,

$$\mu\left\{x:\ \left|n(t,x)-\frac{t}{\bar{l}}\right| > L\sqrt{t}\right\} = \mu\left\{x:\ \sum_{i=0}^{[t/\bar{l}+L\sqrt{t}]} l(\phi^{-i}x) < t\right\} + $$

$$+\ \mu\left\{x:\ \sum_{i=0}^{[t/\bar{l}-L\sqrt{t}]} l(\phi^{-i}x) \geqq t\right\}.$$

Applying Chebyshev's inequality and Lemma 3.2, we get

$$\mu\left\{x:\ \sum_{i=0}^{[t/\bar{l}+L\sqrt{t}]} l(\phi^{-i}x) < t\right\} = \mu\left\{x:\ \sum_{i=0}^{[t/\bar{l}+L\sqrt{t}]} (l(\phi^{-i}x)-\bar{l}) < -L\bar{l}\sqrt{t}\right\}$$

$$\leqq \frac{C_r(t/\bar{l}+L\sqrt{t})^r}{L^{2r}\bar{l}^{2r}t^r} \leqq \frac{C_r'}{2L^r}$$

for sufficiently large $t>0$, where $C_r'>0$ is a constant. Then, for large t and all $r>0$,

$$\mu\left\{x:\ \left|n(t,x)-\frac{t}{\bar{l}}\right| > L\sqrt{t}\right\} \leqq \frac{C_r'}{L^r}. \tag{18}$$

Proof of Theorem 3.1. Using the notation of Section 2, we consider $a(t,w)$ and $B(t,x)$, $w=(x,y)$. We have

$$\left|\int_W a^2(t,w)dv - \int_X B^2(t,x)d\mu_l\right| \leqq \frac{C_1}{\sqrt{t}}\int_X B^2(t,x)d\mu_l + \frac{C_1^2}{t} \text{ and}$$

$$\left|\int_X B^2(t,x)d\mu_l - \int_X \frac{D^2(t,x)}{t}d\mu_l\right| \leqq \frac{C_1}{\sqrt{t}}\int_X \frac{D^2(t,x)}{t}d\mu_l + \frac{C_1^2}{t} \tag{19}$$

where

$$D(t,x) = \sum_{i=0}^{n(t,x)}\left[F(\phi^{-i}x) - \frac{\bar{F}}{\bar{l}}l(\phi^{-i}x)\right] = \sum_{i=0}^{n(t,x)}\tilde{F}(\phi^{-i}x).$$

On the set A_{kt}:

$$\left|D(t,x)/\sqrt{t} - \sum_{i=0}^{[t/\bar{l}+k\varepsilon\sqrt{t}]}\tilde{F}(\phi^{-i}x)/\sqrt{t}\right| \leqq R\varepsilon$$

where $R > 0$ is a constant. Let us denote

$$S_k(t,x) = \frac{1}{\sqrt{t}} \sum_{i=0}^{[t/l + k\varepsilon\sqrt{t}]} \tilde{F}(\phi^{-i}x) \text{ and}$$

$$S(t,x) = S_k(t,x) \text{ for } x \in A_{kt}.$$

Then

$$(20) \qquad \left| \int_X \frac{D^2(t,x)}{t} d\mu_l - \int_X S^2(t,x) d\mu_l \right| \leqq R\varepsilon \int_X S^2(t,x) d\mu_l + R^2\varepsilon^2.$$

We now define a function $h_N(y)$, continuous on $-\infty < y < \infty$:

$$h_N(y) = \begin{cases} 1, \text{ for } |y| \leqq N, \\ 0 \leqq h_N(y) \leqq 1 \text{ on } [-N-1, -N] \text{ and } [N, N+1], \\ 0, \text{ for } |y| \geqq N+1. \end{cases}$$

Then

$$(21) \quad E_{\mu_l}[S^2(t,x)] = E_{\mu_l}[S^2(t,x) \cdot h_N(S(t,x))] + E_{\mu_l}[S^2(t,x)(1 - h_N(S(t,x)))].$$

The function $z^2 h_N(z)$ is bounded and continuous for fixed N. It, therefore follows from Theorem 2.1 that

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} z^2 h_N(z) d\Phi_t(z) = \int_{-\infty}^{\infty} z^2 h_N(z) \exp(-z^2/2\sigma^2) dz$$

where $\Phi_t(z)$ is the distribution of $S(t,x)$ and $\sigma^2 = (2\pi/l)\, r_{F(x)}(0)$.

The second term in (21) satisfies the estimate

$$E_{\mu_l}[S^2(t,x)(1 - h_N(S(t,x))] \leqq E_{\mu_l}[S^2(t,x)\chi\, |S(t,x)| > N]$$

where χ_A denotes the indicator function of the set A.

Denote

$$\xi(t,x) = (n(t,x) - t/l)/\sqrt{t}.$$

Then the set A_{kt} may be expressed as

$$A_{kt} = \{x: k\varepsilon \leqq \xi(t,x) < (k+1)\varepsilon\}.$$

Applying the Schwartz inequality, Chebyshev's inequality and Lemma 3.3, we obtain

$$E_{\mu_l}[S^2(t,x)\chi\,|S(t,x)| > N] = \sum_k \int_{[k\varepsilon \leqq \xi \leqq (k+1)\varepsilon]} [S_k(t,x)]^2 \chi\, |S(t,x)| > N d\mu_l$$

$$(22) \quad \begin{aligned} &\leqq \sum_k \left[\int_{[k\varepsilon \leqq \xi \leqq (k+1)\varepsilon]} d\mu_l \right]^{\frac{1}{2}} \cdot \left[\int_{[k\varepsilon \leqq \xi \leqq (k+1)\varepsilon]} [S_k(t,x)]^4 \chi |S(t,x)| > N d\mu_l \right]^{\frac{1}{2}} \\ &\leqq Q \cdot \sum_k \left[\frac{E_{\mu_l}(\xi(t,x))^{2s}}{(\widetilde{k\varepsilon})^{2s}} \right]^{\frac{1}{2}} \cdot \left[\frac{E_{\mu_l}(S_k(t,x))^{2m}}{N^{2m-4}} \right]^{\frac{1}{2}} \end{aligned}$$

where s, $m > 0$ are integers, $m \geqq 2$, $Q > 0$ is a constant, and

$$\frac{E_{\mu_t}(\xi(t,x))^{2s}}{(\tilde{k}\varepsilon)^{2s}} = \begin{cases} \dfrac{E_{\mu_t}(\xi(t,x))^{2s}}{(k\varepsilon)^{2s}} & k > 0 \\ 1 & k = 0, -1 \\ \dfrac{E_{\mu_t}(\xi(t,x))^{2s}}{|(k+1)\varepsilon|^{2s}} & k < -1. \end{cases}$$

By Lemma 3.2, the following inequality holds on A_{kt}:

$$E_{\mu_t}(S_k(t,x))^{2m} \leqq C_m\left[\frac{1}{l} + \frac{k\varepsilon}{\sqrt{t}}\right]^m \leqq C_m\left[\frac{1}{l} + \frac{\xi(t,x)}{\sqrt{t}}\right]^m.$$

But

$$\frac{n(t,x)}{t} = \frac{1}{l} + \frac{\xi(t,x)}{\sqrt{t}} < \frac{t/L_1}{t} = \frac{1}{L_1}$$

where L_1 is such that $l(x) \geqq L_1 > 0$. Therefore,

$$E_{\mu_t}(S_k(t,x))^{2m} \leqq C_m \frac{1}{L_1^m}. \tag{23}$$

It follows from (18) and Lemma 3.3 that $\xi(t,x)$ has finite and bounded moments of any order with respect to t. Thus, setting $s \geqq 2$ and $m \geqq 3$ in (21) and taking account of (23), we see that for all sufficiently large t

$$E_{\mu_t}(S^2(t,x) \,.\, \chi \,|\, S(t,x)| \leqq N) \leqq \tilde{Q}/N$$

where $\tilde{Q} > 0$ is a constant.

It now follows from (21) that

$$\overline{\lim_{t\to\infty}} \left| E_{\mu_t}(S^2(t,x)) - \int_{-\infty}^{\infty} z^2 h_N(z) \exp(-z^2/2\sigma^2)\,dz \right| \leqq \tilde{Q}/N$$

and, since N is arbitrary,

$$\lim_{t\to\infty} E_{\mu_t}(S^2(t,x)) = \sigma^2.$$

Since (20) is valid for any $\varepsilon > 0$, we have

$$\lim_{t\to\infty} E_{\mu_t}\left(\frac{D^2(t,x)}{t}\right) = \sigma^2.$$

The theorem now follows from (19). ∎

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INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL